Modul 3

Finite-difference time-domain (FDTD)

based on Dennis Sullivan, A Brief Introduction to The Finite-Difference Time-Domain (FDTD) Method http://www.mrc.uidaho.edu/~dennis/ECE538-files/Intro(FDTD).doc

Maxwell's equations, formulated circa 1870, represent a fundamental unification of electric and magnetic fields predicting electromagnetic wave phenomena which Nobel Laureate Richard Feynman has called the most outstanding achievement of 19th-century science. Now, engineers and scientists worldwide use computers ranging from simple desktop machines to massively parallel arrays of processors to obtain solutions of these equations. As we begin the 21st century, it may seem a little odd to devote so much effort to study solutions of the 19th century's best equations. Thus we ask the question: "Of what relevance is the study of electromagnetics to our modern society?"

The goal of this unit is to help answer this question. We shall discuss prospects for using numerical solutions of Maxwell's equations, in particular the finite-difference time-domain (FDTD) method, to help innovate and design key electrical engineering technologies ranging from cellphones and computers to lasers and photonic circuits. Whereas the study of electromagnetics has been motivated in the past primarily by the requirements of military defense, the entire field is shifting rapidly toward important commercial applications in high-speed communications and computing that touch everyone in their daily lives. Ultimately, this will favorably impact the economic well-being of nations as well as their military security.

1.1 One-dimensional Simulation in Free Space

Electromagnetics is governed by the time-dependent Maxwell's curl equations, which in free space are

$$\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{\varepsilon_0} \nabla \times \boldsymbol{H} \qquad (1.1\ a)$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = -\frac{1}{\mu_0} \nabla \times \boldsymbol{E} \,. \qquad (1.1\ b)$$

E and ***H*** are vectors in three dimensions, but if we consider only one dimension

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0} \frac{\partial H_y}{\partial z} \qquad (1.2\ a)$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z} \,. \qquad (1.2\ b)$$

To put these equations in a computer, we approximate the derivatives with the "finite-difference" approximations:

$$\frac{E_x^{n+1/2}(k) - E_x^{n-1/2}(k)}{\Delta t} = -\frac{1}{\varepsilon_0} \frac{H_y^n(k+1/2) - H_y^n(k-1/2)}{\Delta x} \qquad (1.3\ a)$$

$$\frac{H_y^{n+1}(k+1/2) - H_y^{n}(k+1/2)}{\Delta t} = -\frac{1}{\mu_0}\frac{E_x^{n+1/2}(k+1) - E_x^{n+1/2}(k)}{\Delta x} \quad . \qquad (1.3\text{ b})$$

In these two equations, time is specified by the superscripts, i. e., "*n*" actually means a time $t = \Delta t \cdot n$, and "k" actually means the distance $z = \Delta x \cdot k$. (It might seem more sensible to use Δz as the incremental step, since in this case we are going in the z direction. However, Δx is so commonly used for a spatial increment that I will use Δx.)
We rearrange the above equations to :

$$E_x^{n+1/2}(k) = E_x^{n-1/2}(k) - \frac{\Delta t}{\varepsilon_0 \cdot \Delta x}\left[H_y^{n}(k+1/2) - H_y^{n}(k-1/2)\right] \qquad (1.4\text{ a})$$

$$H_y^{n+1}(k+1/2) = H_y^{n}(k+1/2) - \frac{\Delta t}{\mu_0 \cdot \Delta x}\left[E_x^{n+1/2}(k+1) - E_x^{n+1/2}(k)\right]. \qquad (1.4\text{ b})$$

Notice that the calculations are interleaved in both space and time. In Eq. (1.4 a), for example, the new value of E_x is calculated from the previous value of E_x and the most recent values of H_y. This is the fundamental paradigm of the finite-difference time-domain (FDTD) method Fig. 1.1) [1].

Eq. (1.4 a) and (1.4 b) look very similar. However, ε_0 and μ_0 differ by several orders of magnitude:

$$\varepsilon_0 = 8.85\times 10^{-12}\ F/m,$$
$$\mu_0 = 4\pi\times 10^{-7}\ H/m.$$

Therefore, E_x and H_y will differ by several orders of magnitude. This is circumvented by making the following change of variables [2]:

$$\tilde{E} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,E. \qquad (1.5)$$

Substituting this into Eq. (1.4a) and (1.4b) gives

$$\tilde{E}_x^{n+1/2}(k) = \tilde{E}_x^{n-1/2}(k) - \frac{1}{\sqrt{\varepsilon_0\mu_0}}\frac{\Delta t}{\Delta x}\left[H_y^{n}(k+1/2) - H_y^{n}(k-1/2)\right] \qquad (1.6\text{a})$$

$$H_y^{n+1}(k+1/2) = H_y^{n}(k+1/2) - \frac{1}{\sqrt{\varepsilon_0\mu_0}}\frac{\Delta t}{\Delta x}\left[\tilde{E}_x^{n+1/2}(k+1) - \tilde{E}_x^{n+1/2}(k)\right] \qquad (1.6\text{b})$$

Now both $\tilde{E}$ and H will have the same order of magnitude. We will call this "normalized" units. Physicist call this Gaussian units. Note that

$$\left[\sqrt{\frac{\varepsilon_0}{\mu_0}}\right] = \left[\frac{F/m}{H/m}\right]^{1/2} = \left[\frac{C/V}{Wb/A}\right]^{1/2}$$
$$= \left[\frac{C/V}{V\cdot s/A}\right]^{1/2} = \left[\frac{A}{V}\right] = \left[\frac{1}{\Omega}\right]$$

and

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi\times 10^{-7}}{8.85\times 10^{-12}}} = \sqrt{1.42\times 10^{5}} = 377\ \Omega.$$

This quantity is called the "impedance of free space."

Once the cell size Δx is chosen, then the time step Δt is determined by

$$\Delta t = \frac{\Delta x}{2 \cdot c_0} \qquad (1.7)$$

where c_0 is the speed of light in free space. Therefore,

$$\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \cdot \frac{\Delta t}{\Delta x} = c_0 \cdot \frac{\Delta x / 2 \cdot c_0}{\Delta x} = \frac{1}{2} \qquad (1.8)$$

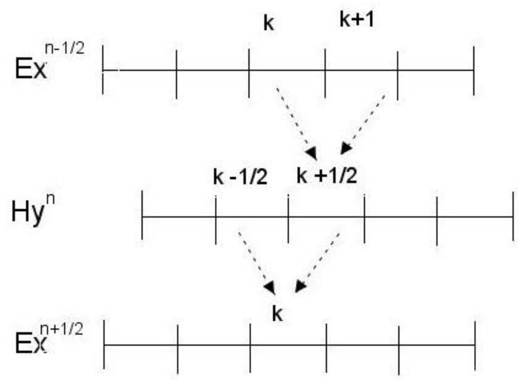


Figure 1.1. A diagram of the calculation of E and H fields in FDTD.

Re-writing Eq. (1.6 a) and (1.6 b) in C computer code gives the following:

```
ex[k] = ex[k] + 0.5*( hy[k-1] - hy[k] )
```
(1.9 a)

```
hy[k] = hy[k] + 0.5*( ex[k] - ex[k+1] )
```
(1.9 b)

Note that the *n* or *n+1/2* or *n-1/2* in the superscripts is gone. Time is implicit in the FDTD method. In Eq. (1.9 a), the `ex` on the right side of the equal sign is the previous value at *n - 1/2*, and the `ex` on the left side is the new value, *n+1/2*, which is being calculated. Position, however, is explicit. The only difference is that *k + 1/2* and *k - 1/2* are rounded off to *k* and *k-1* in order to specify a position in an array in the program. Figure 1.2 illustrates a simulation in free space. The following things are worth noting:

1. The E_x and H_y values are calculated by separate loops, and they employ the interleaving described above.
2. After the E_x values are calculated, the source is calculated. This is done by simply specifying a value of E_x at the point `k = 1`, and overriding what was previously calculated. This is referred to as a "hard source," because a specific value is imposed on the FDTD grid.

Simulation in a Lossless Dielectric Material.

Now consider the case where the medium is not free space but a medium that has a relative dielectric constant other than one. That mean Eq. (1.2 a) must be written

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_r \varepsilon_0} \frac{\partial H_y}{\partial x}.$$

{In this class we will not be dealing with magnetic material, so the permeability is always $\mu = \mu_0$. Therefore, Eq. (1.2 b) does not change.} If we got through the same finite-difference approximation and switch to normalized units, Eq (1.6 a) becomes

```
ex[k] = ex[k] + cb[k]*( hy[k-1] - hy[k] ),
```

cb[k]=0.5/epsilon.

1.2 Simulation in a Lossy medium

Once more we will start with the time-dependent Maxwell's curl equations, but we will write them in a more general form, which will allow us to simulate propagation in media which have conductivity:

$$\varepsilon \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H} - \boldsymbol{J} \qquad (1.10\text{ a})$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = -\frac{1}{\mu_0} \nabla \times \boldsymbol{E} . \qquad (1.10\text{ b})$$

$\boldsymbol{J}$ is the current density, which can also be written

$$\boldsymbol{J} = \sigma \cdot \boldsymbol{E} ,$$

where σ is the conductivity. Putting this into Eq. (1.10 a) and dividing through by the dielectric constant we get

$$\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \nabla \times \boldsymbol{H} - \frac{\sigma}{\varepsilon_0 \varepsilon_r} \boldsymbol{E} .$$

We now revert to our simple one-dimensional equation:

$$\frac{\partial E_x(t)}{\partial t} = \frac{1}{\varepsilon_r \varepsilon_0} \cdot \frac{\partial H_y(t)}{\partial z} - \frac{\sigma}{\varepsilon_r \varepsilon_0} E_x(t) ,$$

and make the change of variable in Eq. (1.5) which gives

$$\frac{\partial \tilde{E}_x(t)}{\partial t} = \frac{1}{\varepsilon_r \cdot \sqrt{\varepsilon_0 \mu_0}} \cdot \frac{\partial H_y(t)}{\partial z} - \frac{\sigma}{\varepsilon_r \varepsilon_0} \tilde{E}_x(t) \qquad (1.11\text{ a})$$

$$\frac{\partial H_y(t)}{\partial t} = -\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \cdot \frac{\partial \tilde{E}_x(t)}{\partial z} . \qquad (1.11\text{ b})$$

Next take the finite difference approximations for both the temporal and spatial derivatives similar to Eq. (1.3 a):

$$\begin{aligned} \frac{\tilde{E}_x^{n+1/2}(k) - \tilde{E}_x^{n-1/2}(k)}{\Delta t} &= \frac{1}{\varepsilon_r \cdot \sqrt{\varepsilon_0 \mu_0}} \cdot \frac{H_y^n(k+1/2) - H_y^n(k-1/2)}{\Delta x} \\ &\quad - \frac{\sigma}{\varepsilon_r \varepsilon_0} \frac{\tilde{E}_x^{n+1/2}(k) + \tilde{E}_x^{n-1/2}(k)}{2} . \end{aligned} \qquad (1.12)$$

Notice that the last term in Eq. (1.11 a) is approximated as the average across two time steps in Eq. (1.12). From the previous section

$$\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \frac{\Delta t}{\Delta x} = \frac{1}{2} ,$$

so Eq. (1.12) becomes

$$\tilde{E}_x^{n+1/2}(k)\left[1+\frac{\Delta t\cdot\sigma}{2\varepsilon_r\varepsilon_0}\right]=\tilde{E}_x^{n-1/2}(k)\left[1-\frac{\Delta t\cdot\sigma}{2\varepsilon_r\varepsilon_0}\right]$$
$$+\frac{1/2}{\varepsilon_r}\left[H_y^n(k+1/2)-H_y^n(k-1/2)\right]$$

or

$$\tilde{E}_x^{n+1/2}(k)=\frac{\left(1-\dfrac{\Delta t\cdot\sigma}{2\varepsilon_r\varepsilon_0}\right)}{\left(1+\dfrac{\Delta t\cdot\sigma}{2\varepsilon_r\varepsilon_0}\right)}\tilde{E}_x^{n-1/2}(k)+\frac{1/2}{\varepsilon_r\cdot\left(1+\dfrac{\Delta t\cdot\sigma}{2\varepsilon_r\varepsilon_0}\right)}\left[H_y^n(k+1/2)-H_y^n(k-1/2)\right].$$

From these we can get the computer equations

```
ex[k] = ca[k]*ex[k] + cb[k] *( hy[k-1] - hy[k] )    (1.13 a)
hy[k] = hy[k] + 0.5*( ex[k] - ex[k+1] ),         (1.13 b)
```

where

```
eaf = dt*sigma/(2*epsz*epsilon)        (1.14 a)
ca[k] = (1. - eaf)/(1. + eaf)  (1.14 b)
cb[k] =0.5/(epsilon*(1. + eaf)).     (1.14 c)
```

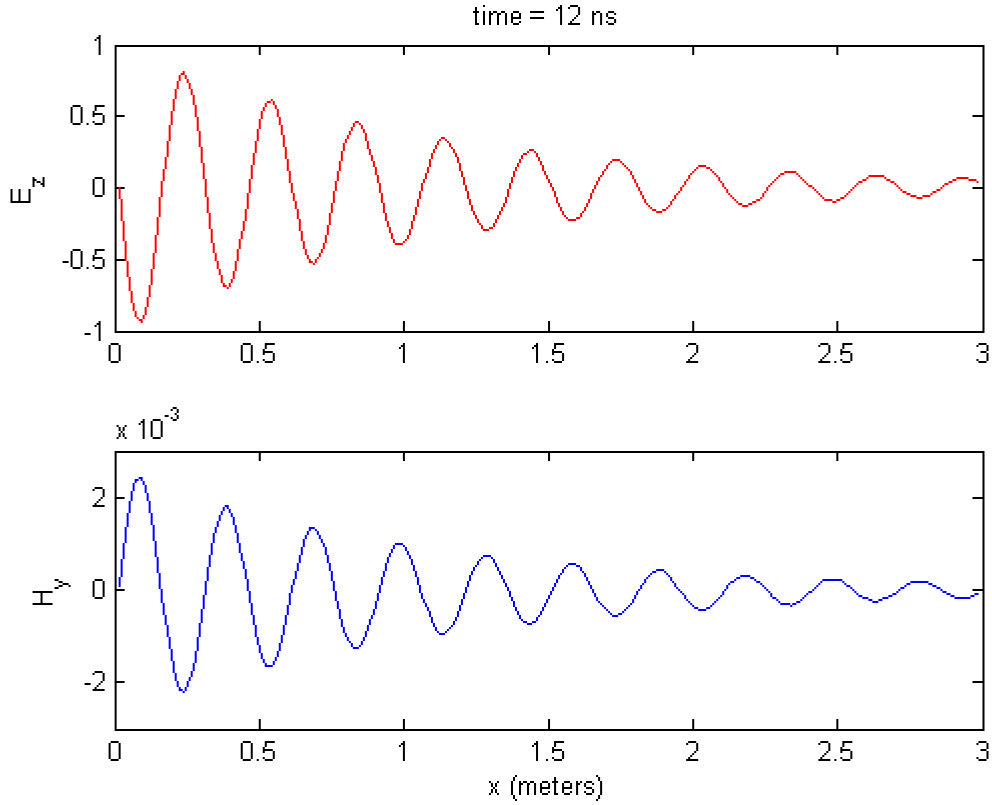


Figure 1.1 Simulation (source code in fdtd1d.m) of 1-GHz sinusoidal wave propagating in a nonpermeable lossy medium (epsr=1.0, sigma=5.0e-3 S/m). The grid resolution (dx = 1.5 cm) is chosen to provide 20 samples per wavelength. The Courant factor S=c*dt/dx is set to the stability limit (S=1). In 1-D, this is the "magic time step." The total number of time steps (nmax=240) corresponds to a physical time of 12 ns. The grid is terminated with electric-field components at the far-left (i=1) and far-right (i=ie) boundaries. The sinusoidal wave is launched by an electric-field hard-source condition at i=1. The simplest radiation boundary condition for plane wave propagation is used to update the electric field at i=ie: Ez(ie,n+1) = Ez(ie-1,n).

1.3 Courant condition and numerical stability

An electromagnetic wave propagating in free space cannot go faster than the speed of light. To propagate a distance of one cell requires a minimum time dt=dx/c. Courant condition imposes that the Courant factor S=c*dt/dx must be smaller than the limit S=1. In n dimensions the stability is achieved for $S<1/\sqrt{n}$. In relations 1.7,1.8 was chosen Courant factor S=1/2.

1.4 Numerical dispersion

The numerical algorithm causes the dispersion of the simulated wave modes in the computational space. That is, the phase velocity of numerical modes can vary with wavelength, direction of propagation and lattice discretization. This numerical dispersion can lead to non-physical results and must be taken in account to understand the FDTD algorithm and its accuracy limits.

In 1D the dispersion relation is:

$$(1/c\Delta t)^2 \sin^2(\omega\Delta t/2)=(1/\Delta x)^2 \sin^2(k\Delta x/2)$$

In the limit when Δx, Δt tend to zero it reduces to dispersion relation in continuum medium:

$$\omega^2=k^2c^2$$

1.5 Absorbing boundary condition in 1D

Absorbing boundary conditions are necessary to keep outgoing E and H fields from being reflected back. In order to calculate E field we need to know the surrounding H values. But at the edge we will not have the value to one side. But we know that there are no sources outside the simulation space and the fields at the edge must be propagating outward.
If Courant factor is S=1/2 dt=dx/2 and we need two time steps for a wave front to cross one cell.
An acceptable boundary condition might be $Ex^{n}(-0) = Ex^{n-2}(+0)$.
For S=1 the boundary condition is $Ex^{n}(-0) = Ex^{n-1}(+0)$.

Perfect magnetic conductor (PMC). The magnetic-field node in the grid is initially zero and remains zero throughout the simulation. When the field encounters this node it essentially see a perfect magnetic conductor (PMC). To satisfy the boundary condition at this node, i.e., that the total magnetic field go to zero, a reflected wave is created which reverses the sign of the magnetic field but preserves the sign of the electric field.

Perfect electric conductor (PEC). The electric-field node in the grid is initially zero and remains zero throughout the simulation. When the field encounters this node it behaves like a perfect electric conductor (PEC). To satisfy the boundary conditions at this node, the wave is again reflected, but this time the electric field changes sign while the sign of the magnetic field is preserved.

1.6 Sources

In simulation we have hard and soft sources. We call a hard source when a value is assigned to Ex. Ex is usually dcalculated at some point by the evolution equation. For a hard source we are specifying the desired value of Ex and overriding what was previously calculated. A propagating pulse will see that value and be reflected.
We call a soft source when lue is added to Ex at a certain point. A propagating pulse will just pass through a soft source.
We can have different kinds of sources:
sinusoidal wave: sin(omega*t)
temporal Gaussian: exp(-t*t/2)
1.7 Cell size

Enough sampling points must be taken to ensure that an adequate representation of continuum is made. A good rule of thumb is 10 points per wavelength (for the medium with highest dielectric constant because it corresponds to the medium with the shortest wavelength in the simulation space).

2. 2D FDTD

Not surprisingly, we will start with Maxwell's equations

$$\frac{\partial \tilde{D}}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \cdot \nabla \times H \quad (2.1\ a)$$

$$\tilde{D}(\omega) = \varepsilon_r^{*}(\omega) \cdot \tilde{E}(\omega) \quad (2.1\ b)$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \nabla \times \tilde{E} \quad (2.1\text{ c})$$

Once again, we will drop the ~ notation, but it will always be assumed that we are referring to the normalized values.

Eqs. (2.1.a) and (2.1.c) produce six scalar equations, that ca be grouped in 2 sets of equations: transverse-magnetic (TM z-polarized) mode equations containing Hx, Hy and Ez and transverse-electric (TE z-polarized) mode equations containing Ex, Ey and Hz. The TM equations are:

$$\frac{\partial D_z}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \quad (2.2\text{ a})$$

$$D_z(\omega) = \varepsilon_r^*(\omega) \cdot E_z(\omega) \quad (2.2\text{ b})$$

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \frac{\partial E_z}{\partial y} \quad (2.2\text{ c})$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \frac{\partial E_z}{\partial x}. \quad (2.2\text{ d})$$

The finite difference equations are:

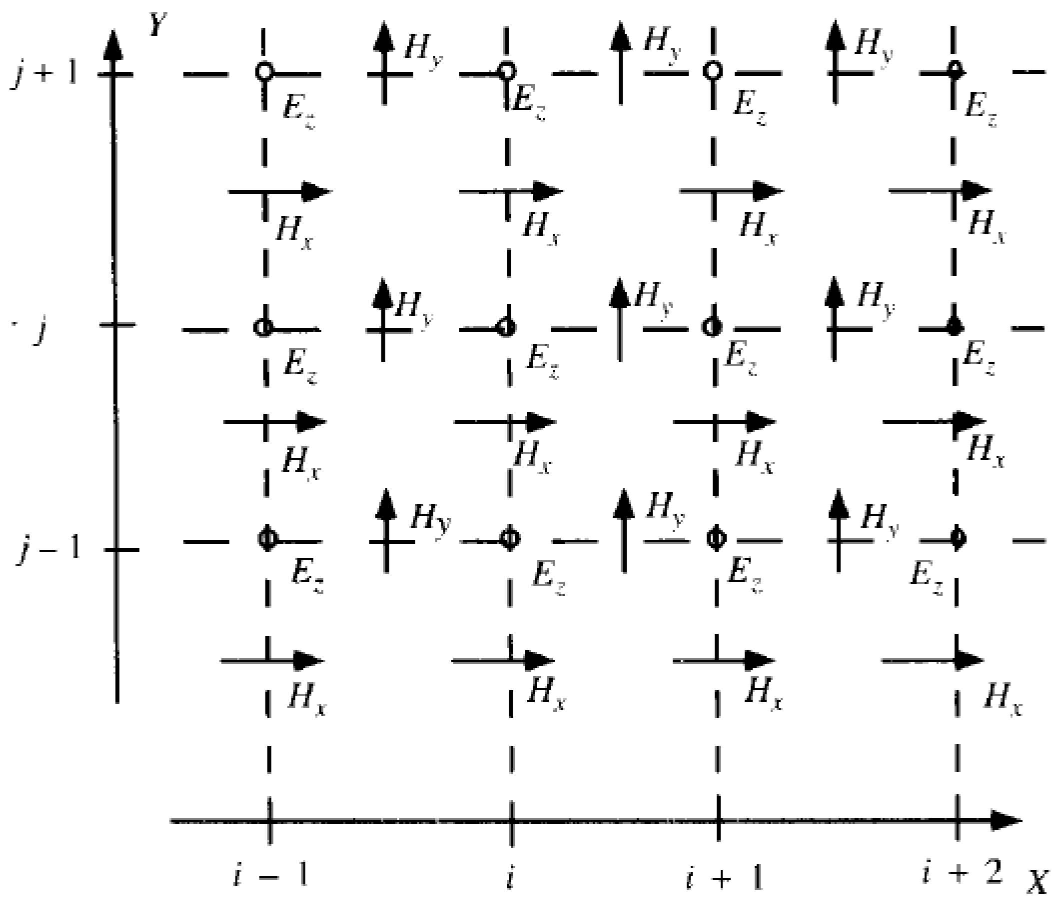
$$\frac{D_z^{n+1/2}(i, j) - D_z^{n-1/2}(i, j)}{\Delta t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \left(\frac{H_y^n(i + 1/2, j) - H_y^n(i - 1/2, j)}{\Delta x} \right) - \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \left(\frac{H_x^n(i, j + 1/2) - H_x^n(i, j - 1/2)}{\Delta x} \right)$$

$$\frac{H_x^{n+1}(i, j + 1/2) - H_x^n(i, j + 1/2)}{\Delta t} = -\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \frac{E_z^{n+1/2}(i, j + 1) - E_z^{n+1/2}(i, j)}{\Delta x}$$

$$\frac{H_y^{n+1}(i + 1/2, j) - H_y^n(i + 1/2, j)}{\Delta t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \frac{E_z^{n+1/2}(i + 1, j) - E_z^{n+1/2}(i, j)}{\Delta x}.$$

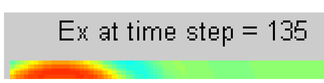
(2.3)

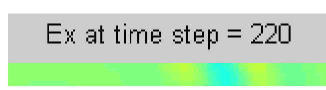
Where we used the following interleave of E and H fields:



The TE equations can be obtained by permuting E and H fields in eq. (2.2) and (2.3). Taking the time step Δt :

$$\Delta t = \frac{\Delta x}{2 \cdot c_0} \qquad (2.4)$$

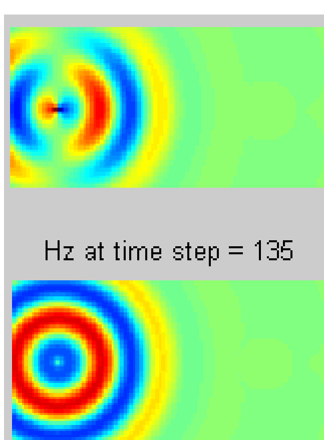




we get the C code:

```
dz[i][j] = dz[i][j]
        +.5*(hy[i][j] - hy[i-1][j] - hx[i][j] + hx[i][j-1]);
ez[i][j] = gaz[i][j]*(dz[i][j] - iz[i][j] );
iz[i][j] = iz[i][j] + gbz[i][j]*ez[i][j];
hx[i][j] = hx[i][j] + .5*( ez[i][j] - ez[i][j+1] );
hy[i][j] = hy[i][j] + .5*( ez[i+1][j] - ez[i][j] );
```

(2.5 a)
(2.5 b)
(2.5 c)
(2.5 d)
(2.5 e)



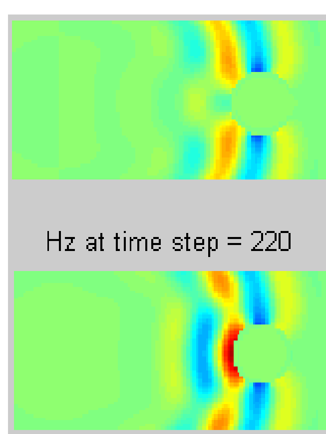


Figure13.6. Simulation (source code in fdtd2d.m) of air-filled rectangular cavity resonator with a 6cm metal cylindrical scatterer. The cavity is excited by a Gaussian pulse with a carrier frequency of 5 GHz.The grid resolution (dx = 3 mm) was chosen to provide at least 20 samples per wavelength at the center frequency of the pulse (which in turn provides approximately 10 samples per wavelength at the high end of the excitation spectrum, around 10 GHz).The computational domain is truncated using the perfectly matched layer (PML) absorbing boundary conditions. The formulation used in this code is based on the original split-field Berenger PML. Exponential time stepping is implemented in the PML regions.

3. 3D FDTD

3.1 3D FDTD

The original FDTD paradigm was described by the "Yee Cell," (Fig. 3.1), named, of course, after Kane Yee [1]. Note that the *E* and *H* fields are assumed interleaved around a cell whose origin is at the location I, J, K. Every *E* field is located 1/2 cell width from the origin in the direction of its orientation; every *H* field is offset 1/2 cell in each direction except that of its orientation.

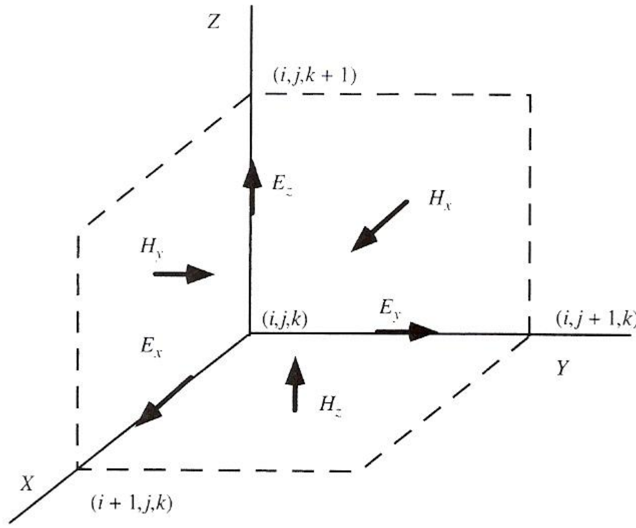


Figure 3.1. The Yee cell.

Not surprisingly, we will start with Maxwell's equations

$$\frac{\partial \tilde{D}}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \cdot \nabla \times H \quad (3.1\ a)$$

$$\tilde{D}(\omega) = \varepsilon_r^*(\omega) \cdot \tilde{E}(\omega) \quad (3.1\ b)$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \nabla \times \tilde{E} \,. \quad (3.1\ c)$$

Once again, we will drop the ~ notation, but it will always be assumed that we are referring to the normalized values.
Eqs. (3.1.a) and (3.1.c) produce six scalar equations, two of which are:

$$\frac{\partial D_z}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \quad (3.2\ a)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} \right) . \quad (3.2\ a)$$

The first step is to take the finite difference approximations.

$$\begin{aligned} D_z^{n+1/2}(i,j,k+1/2) &= D_z^{n-1/2}(i,j,k+1/2) \\ &+ \frac{\Delta t}{\Delta x \cdot \sqrt{\varepsilon_0 \mu_0}} (H_y^n(i+1/2,j,k+1/2) - H_y^n(i-1/2,j,k+1/2) \\ &- H_x^n(i,j+1/2,k+1/2) + H_x^n(i,j-1/2,k+1/2)) \end{aligned} \quad (3.3\ a)$$

$$\begin{aligned} H_z^{n+1}(i+1/2,j+1/2,k) &= H_z^n(i+1/2,j+1/2,k) \\ &+ \frac{\Delta t}{\Delta x \cdot \sqrt{\varepsilon_0 \mu_0}} (E_y^{n+1/2}(i+1,j+1/2,k) - E_y^{n+1/2}(i,j+1/2,k) \\ &- E_x^{n+1/2}(i+1/2,j+1,k) + E_x^{n+1/2}(i+1/2,j,k)) \end{aligned} \quad . \quad (3.3\ b)$$

The relationship between E and D, corresponding to Eq. (3.1 b) is exactly the same as the one-dimensional or two-dimensional cases, except now there will be three equations. Different materials can be specified at each cell within the FDTD program (Fig. 3.2).

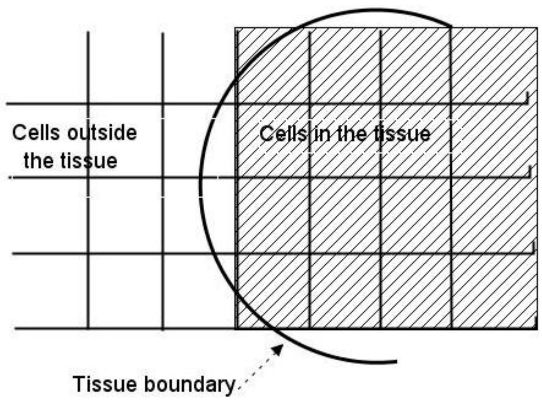


Figure 3.2. Different properties can be specified for each cell in an FDTD program.

Many simulations model the applicator as well as the body being radiated. A simple dipole antenna is illustrated in Fig. 3.3. It consists of two metal arms. A dipole antenna functions by having current run through the arms, which results in radiation. FDTD simulates a dipole in the following way: The metal of the arms is specified by setting the appropriate parameters to zero in the cells corresponding to metal. This insures that the corresponding E_z field at this point remains zero, as it would if that point were inside metal. The source is specified by setting the E_z field in the gap to a certain value. (In a real dipole antenna, the E_z field in the gap would be the result of the current running through the metal arms.) Notice that we could have specified a current in the following manner: Ampere's circuital law says

$$\oint_C \boldsymbol{H} \cdot dl = I ,$$

ie., the current through a surface is equal to the line integral of the H field around the surface.

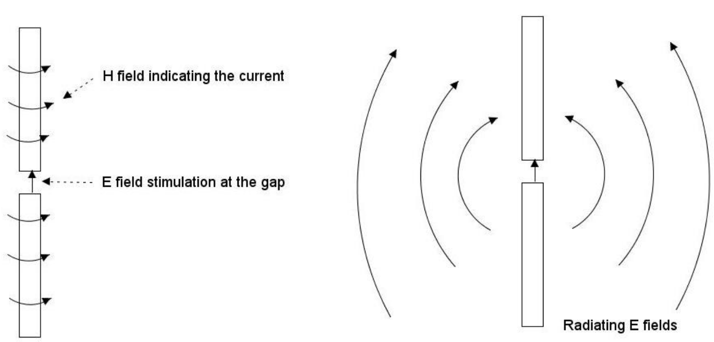


Figure 3.3. A dipole antenna is simulated by specifying the cells of the antenna arms with values that insure the E field will remain at zero. The input stimulation is accomplished by specifying an E field at the gap of the dipole. The current in the dipole arms is simulated by the surrounding H fields. The E fields will radiate outward.

3.2 The Perfectly Matched Layer (PML)

Without the proper truncation of the problem space by appropriate boundary conditions, unwanted reflections would return to cause errors in the simulation (Fig. 3.4). These outgoing waves must be eliminated by an absorbing boundary condition (ABC).

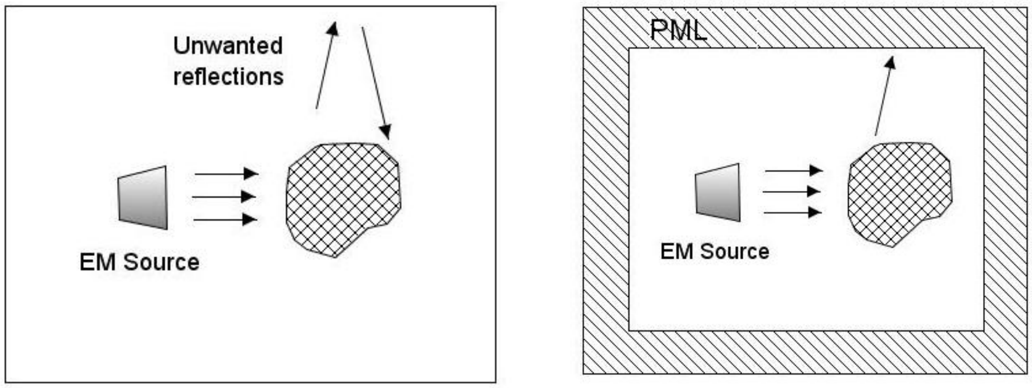


Figure 3.4. Without an absorbing boundary condition, outgoing waves would be reflected back into the problem space (left). The perfectly matched layer (PML) is one of the best means of truncating outgoing waves (right).

One of the most flexible and efficient ABCs is the perfectly matched layer (PML) developed by Berenger [2]. The basic idea is this: if a wave is propagating in medium A and it impinges upon medium B, the amount of reflection is dictated by the intrinsic impedances of the two media (Fig. 3.5)

$$\Gamma = \frac{\eta_A - \eta_B}{\eta_A + \eta_B}, \quad (3.4)$$

which are determined by the dielectric constants ε and permeabilities μ of the two media

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}. \qquad (3.5)$$

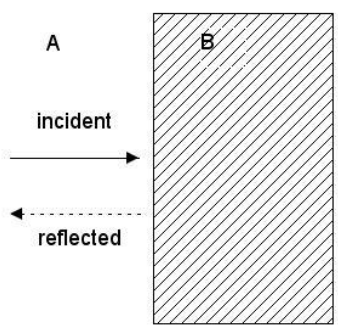


Figure 3.5 The reflection at the interface of two media is dependent on their respective impedances.

Normally, in free space, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$, but in our normalized units, $\eta = 1$ in free space. When we added the flux density formulation, we switched to

$$\varepsilon_F \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H}$$
$$\mathbf{D}(\omega) = \varepsilon^*(\omega)\mathbf{E}(\omega)$$
$$\mu_F \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}$$

We have added "fictitious" dielectric constant and permeability ε_F, μ_F that we will use to implement the PML.
Sacks, et al. [3], shows that there are two conditions to form a PML:

1. The impedance going from the background medium to the PML must be constant,

$$\eta_0 = \eta_m = \sqrt{\frac{\mu^*_{Fx}}{\varepsilon^*_{Fx}}} = 1. \quad (3.6)$$

The impedance is one because of our normalized units.

2. In the direction perpendicular to the boundary (the x direction, for instance), the relative dielectric constant and relative permeablitiy must be the inverse of those in the other directions, i.e.,

$$\varepsilon^*_{Fx} = \frac{1}{\varepsilon^*_{Fy}} \quad (3.7\text{ a})$$
$$\mu^*_{Fx} = \frac{1}{\mu^*_{Fy}} \quad (3.7\text{ b})$$

We will assume that each of these is a complex quantity of the form

$$\varepsilon^*_{Fm} = \varepsilon_{Fm} + \frac{\sigma_{Dm}}{j\omega\varepsilon_0} \quad \text{for m = x or y} \quad (3.8\text{ a})$$
$$\mu^*_{Fm} = \mu_{Fm} + \frac{\sigma_{Hm}}{j\omega\mu_0} \quad \text{for m = x or y} \quad (3.8\text{ b})$$

The following selection of parameters satisfies Eqs. (3.7 a) and (3.7 b) [4]:

$$\varepsilon_{Fm} = \mu_{Fm} = 1 \quad (3.9\text{ a})$$
$$\frac{\sigma_{Dm}}{\varepsilon_0} = \frac{\sigma_{Hm}}{\mu_0} = \frac{\sigma_D}{\varepsilon_0}. \quad (3.9\text{ b})$$

Substituting Eq. (3.9) into (3.3), the value in Eq. (3.9) becomes

$$\eta_0 = \eta_m = \sqrt{\frac{\mu^*_{Fx}}{\varepsilon^*_{Fx}}} = \sqrt{\frac{1 + \sigma(x)/j\omega\varepsilon_0}{1 + \sigma(x)/j\omega\varepsilon_0}} = 1.$$

This fulfills the first requirement above. If σ increases gradually as it goes into the PML, causing D_z and H_y to be attenuated.

PML in X direction for two-dimensions

We take eqs.(2.2) and go to Fourier domain in time (d/dt becomes jω):

$$j\omega D_z \cdot \varepsilon^*_{Fz}(x) = c_0 \cdot \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$j\omega H_x \cdot \mu^*_{Fx}(x) = -c_0 \frac{\partial E_z}{\partial y}$$

$$j\omega H_y \cdot \mu^*_{Fy}(x) = c_0 \frac{\partial E_z}{\partial x},$$

Inserting eqs. (3.8) we get:

$$j\omega\left(1 + \frac{\sigma_D(x)}{j\omega\varepsilon_0}\right) D_z = c_0 \cdot \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \quad \text{(3.10 a)}$$

$$j\omega\left(1 + \frac{\sigma_D(x)}{j\omega\varepsilon_0}\right)^{-1} H_x = -c_0 \frac{\partial E_z}{\partial y} \quad \text{(3.10 b)}$$

$$j\omega\left(1 + \frac{\sigma_D(x)}{j\omega\varepsilon_0}\right) H_y = c_0 \frac{\partial E_z}{\partial x}. \quad \text{(3.10 b)}$$

Eq.(3.10 a) with finite difference approximation:

$$D_z^{n+1/2}(i,j) = gi3(i) \cdot D_z^{n-1/2}(i,j)$$
$$+ gi2(i) \cdot 0.5 \cdot \left[H_y^n(i+1/2,j) - H_y^n(i-1/2,j) - H_x^n(i,j+1/2) - H_y^n(i,j-1/2)\right]$$

(3.11a)

with:

$$gi2(i) = \frac{1}{1 + \sigma_D(i) \cdot \Delta t/(2 \cdot \varepsilon_0)}$$

$$gi3(i) = \frac{1 - \sigma_D(i) \cdot \Delta t/(2 \cdot \varepsilon_0)}{1 + \sigma_D(i) \cdot \Delta t/(2 \cdot \varepsilon_0)}.$$

Eq.(3.10 c) with finite difference approximation:

$$H_y^{n+1}(i+1/2,j) = fi3(i+1/2) \cdot H_y^n(i+1/2,j)$$
$$+ fi2(i+1/2) \cdot 0.5 \cdot \left[E_z^{n+1/2}(i+1,j) - E_z^{n+1/2}(i,j)\right], \quad \text{(3.11c)}$$

with:

$$fi2(i+1/2) = \frac{1}{1 + \sigma_D(i+1/2) \cdot \Delta t/(2 \cdot \varepsilon_0)}$$

$$fi3(i+1/2) = \frac{1 - \sigma_D(i+1/2) \cdot \Delta t/(2 \cdot \varepsilon_0)}{1 + \sigma_D(i+1/2) \cdot \Delta t/(2 \cdot \varepsilon_0)}.$$

Eq.(3.10 b) with finite difference approximation:

$$H_x^{n+1}(i, j+1/2) = H_x^n(i, j+1/2) + \frac{c_0 \cdot \Delta t}{\Delta x} curl\_e \; + \frac{\Delta t \cdot c_0}{\Delta x} \frac{\sigma_D(x) \cdot \Delta t}{\varepsilon_0} I_{Hx}^{n+1/2}(i, j+1/2)$$

$$= H_x^n(i, j+1/2) + \frac{c_0 \cdot \Delta t}{\Delta x} curl\_e \; + \frac{\sigma_D(x) \cdot \Delta t}{2\varepsilon_0} I_{Hx}^{n+1/2}(i, j+1/2).$$

(3.11c)

where:

$$curl\_e = \left[E_z^{n+1/2}(i, j) - E_z^{n+1/2}(i, j+1)\right]$$

$$I_{Hx}^{n+1/2}(i, j+1/2) = I_{Hx}^{n-1/2}(i, j+1/2) + curl\_e$$

$$H_x^{n+1}(i, j+1/2) = H_x^n(i, j+1/2) \; + 0.5 \cdot curl\_e$$

$$+ fi1(i) \cdot I_{Hx}^{n+1/2}(i, j+1/2)$$

$$fi1(i) = \frac{\sigma(i) \cdot \Delta t}{2\varepsilon_0}.$$

Several profiles have been suggested for grading σ(i) of PML. The most successful use is the polynomial variation of the PML loss with depth i inside PML region:

$$xn(i) = .333 * \left(\frac{i}{length\_pml}\right)^3$$

$$i = 1, 2, \ldots, length\_pml$$

$$fi1(i) = xn(i) \qquad (3.12)$$

$$gi2(i) = \left(\frac{1}{1 + xn(i)}\right)$$

$$gi3(i) = \left(\frac{1 - xn(i)}{1 + xn(i)}\right).$$

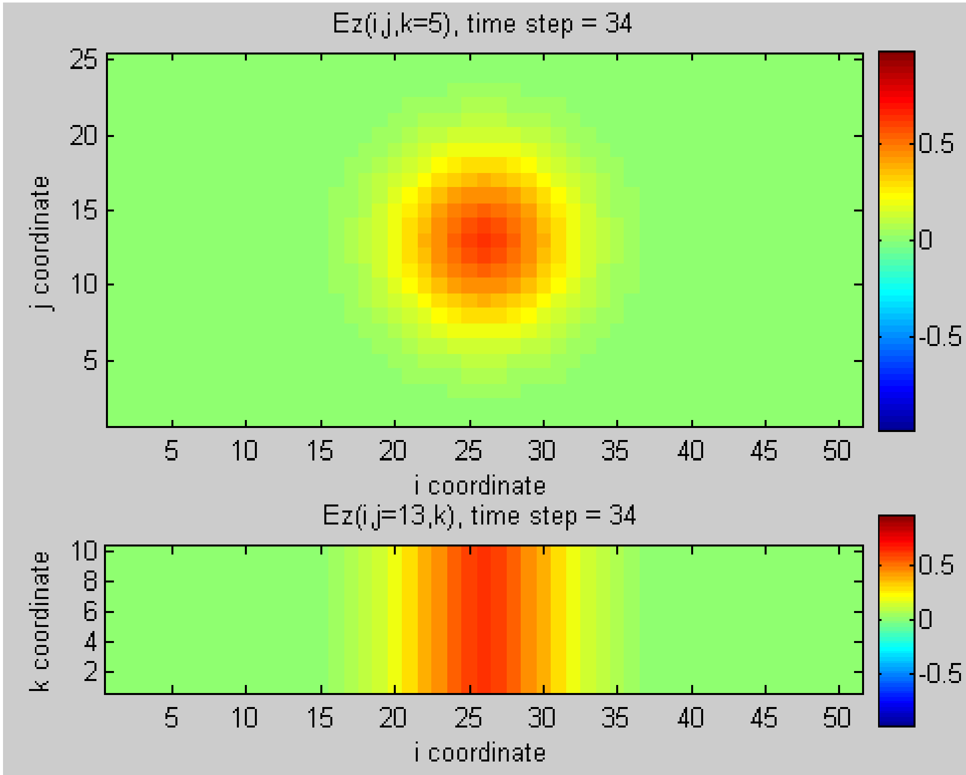


Figure 3.6. Simulation (source code in fdtd3d_pec.m) of air-filled rectangular cavity resonator 10x4.8x2 cm. The cavity is excited by a line of current sources oriented along the z-direction and located in the center of the x-y plane. The source waveform is a differentiated Gaussian pulse given by $J(t)=J0*(t-t0)*\exp(-(t-t0)^2/tau^2)$, where tau=50 ps. The FWHM spectral bandwidth of this zero-dc-content pulse is approximately 7 GHz. The grid resolution (dx = 2 mm) was chosen to provide at least 10 samples per wavelength up through 15 GHz.

References

1.K. S. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Trans. on Antennas and Propagation*, vol. AP-17, pp. 585-589, 1996.
2.J. P. Berenger, “A perfectly matched layer for the absorption of electromagnetic waves*”, J. Comput. Phys.,* vol. 114, pp. 185-200, 1994.
3. Z. S. Sacks, D. M. Kingsland, R. Lee, and J. F. Lee, “A perfectly matched anisotropic absorber for use as an absorbing boundary condition,” *IEEE Trans. Anten. and Prop.,* vol. 43, pp. 1460-1463, Dec. 1995.
4. R. M. Joseph, S. C. Hagness, and A. Taflove, “Direct time integration of Maxwell’s equations in linear dispersive media with absorption for scattering and popagationof femtosecond electromagnetic pulses,” *Optics Letters,* vol. 16, pp. 1412-1414, Sept. 1991.

Books on FDTD

1.A. Taflove, S. C. Hagness, *Computation Electrodynamics: The Finite-Difference Time-Domain Method, 2nd Edition.* Boston, MA: Artech House, 2000.
2.D. M. Sullivan, *Electromagnetic Simulation Using the FDTD Method*. N.Y.: IEEE Press, 2000.

Bibliography

Introduction
Allen Taflove, "Why Study E&M"
http://www.ece.northwestern.edu/ecefaculty/taflove/ WhyStudy.pdf
Allen Taflove, "Advances in FDTD Techniques and Applications in Photonics", http://www.ece.northwestern.edu/ecefaculty/taflove/Photonics_North_talk.pdf
Allen Taflove, "A Perspective on the 40-Year History of FDTD Computational Electrodynamics", http://www.ece.northwestern.edu/ecefaculty/taflove/ACES_talk.pdf

A. Taflove, K.R.Umashankar, FDTD for numerical modeling of electromagnetic wave interactions with arbitrary structures
http://phys.ubbcluj.ro/~evinteler/nanofotonica/FDTD_Taflove.pdf
8.J.R.Nagel, The Finite-Difference Time-Domain (FDTD) Algorithm
http://phys.ubbcluj.ro/~evinteler/FDTD_Nagel.pdf

FDTD programs written by Susan C. Hagness
http://phys.ubbcluj.ro/~evinteler/nanofotonica/programs
http://www.artechhouse.com/Detail.aspx?strBookId=1123

fdtd1D.m - 1-GHz sinusoidal wave propagating in a nonpermeable lossy medium
fdtd2D.m - 6-cm-diameter metal cylindrical scatterer in free space is modeled. The source excitation is a Gaussian pulse with a carrier frequency of 5 GHz
fdtd3D_pec.m - air-filled rectangular cavity resonator with PEC boundary conditions
fdtd3D_UPML.m - electric current source comprised of two collinear Jz components in a cavity with UPML absorbing boundary conditions

Evaluation tests

1. Differential Equations (3D to 1D)
Starting with Maxwell's equations in the time domain (Ampere's and Faraday's laws eq. 1.1), differential form, write the 6 coupled differential equations. (Take the cross products and equate vector components.) Convert these equations to the 1-dimensional TE-to-z case by setting d/dy = d/dz = 0 and Ez = 0. This represents a plane wave propagating in the x-direction. You should end up with equations for Ey and Hz. (The TM-to-z case would have similar equations for Ez and Hy.) Compare the result with eq.1.2.

2. FDTD Equations (1D TE-to-z case)
Convert the 1D TE differential equations above to their FDTD difference form. (Use the central difference formula to approximate the derivatives, and solve for Ey^{n+1} and $Hz^{n+1/2}$.)
Use the 1D FDTD lattice shown below:

```
                 |←----cell I----→|
                 |←----dx-------→|
-------+--------o---------+--------o---------+---------o----------+---------
   Ey(I-1)  Hz(I-1)  Ey(I)    Hz(I)     Ey(I+1)    Hz(I+1)   Ey(I+2) -→I
```

Let the E fields be defined at times n, n+1, n-1, etc.
Let the H fields be defined at times n-1/2, n+1/2, etc.
Compare the result with eq.1.3.

3. Program the FDTD Equations in air (1D TE case)
Modify the source code fdtd1d.m (written for lossy medium) in order to place a forced 2-GHz sinusoidal source in air (sig=0) on Ey at I=inc: Ey(I=inc) = sin(ωt).

```
|-------------------------------|---------------------------------------|
0                             Ey(inc)                                      nx
```

4. Test the FDTD Equations and observe sinusoidal Time Domain Data
Modify the source code fdtd1d.m and use freq = 2GHz, dx = wavelength/20, dt = dx/(2c), nx=120, inc=60. Look at Ey and Hz field at all points as a function of time.
Modify the source code fdtd1d.m and plot the Ey and Hz fields at points A,B,C,D as a function of time for 100 time steps. Give one plot of the four Ey fields, and another of the four Hz fields. Store the Ey fields at point C for use in problem 7.
Plot the Ey field at point D against the analytical value: Ey(x) = sin(ωt-βx), where x is the distance from the source.

```
                                  A  B  C  D
                                Ey(inc)
|----------------------------------|----------------------------------|
0                                inc                                  nx
                                 60                                  120
```

A is located at I=60, at source
B is located at I=63, 3 cells from source
C is located at I=67, 7 cells from source

D is located at I=90, 30 cells from source

5. Observe Pulsed Time Domain Data

Change the source in fdtd1d.m to a raised cosine pulse:

Ey(inc) = 1-cos(ωt) , 0<t<1/Fmax

0 , t >1/Fmax

Use Fmax = 2GHz, dx = wavelength/20, dt=dx/(2c), nx=120, inc=60.

Look at the Ey and Hz fields as a function of time along the mesh.

Plot the Ey fields at points A,B,C,D as a function of time for 100 time steps.

Eliminate the boundary conditions. If you run more than 120 time steps you will see the waves reflect off the ends of the FDTD mesh.

5. Numerical Stability

Test your sinusoidal wave simulations with several values of dt = S* dx / c to verify the stability criterion. For 1D you expect your simulations to become unstable when dt > dx / c (Courant factor S>1).

6. Observe Numerical Dispersion

In fdtd1d.m use the raised cosine pulsed source, Fmax = 2GHz, dt=dx/(2c), nx=220, inc=110. Run for 200 time steps.

Plot the Ey fields as a function of time 30 cells from the source for dx = wavelength/60, wavelength/20, wavelenth/10, and wavelength/5.

Plot the Ey fields 30 cells from the source for the sinusoidal source using dx=wavelength/5 and compare result with the values observed at point D in problem 4.

7. 3D FDTD

Modify source code fdtd3D_pec.m and change boundary conditions from perfect electric conductor (PEC) to perfect magnetic conductor (PMC).

8. Differential Equations (3D to 2D)

Starting with Maxwell's equations in the time domain (Ampere's and Faraday's laws eq. 1.1), differential form, write the 6 coupled differential equations. (Take the cross products and equate vector components.) Convert these equations to the 2-dimensional case by setting d/dz = 0. Group Hx, Hy and Ez equations and obtain transverse-magnetic (TM z-polarized) mode equations. Group Ex, Ey and Hz equations and obtain transverse-electric (TE z-polarized) mode equations.

9. FDTD equations (2D TE z-polarized)

Convert the 2D TE differential equations above to their FDTD difference form. Compare with the update equations for E and H fields in the source code fdtd2D.m.

10. 2D Scatterer

Change in the source code fdtd2D.m the metal cylinder with square (size 6x6 cm) dielectric with dielectric constant 12.

11. 2D PML boundary conditions

Compare eqs. (3.11) with 2D PML conditions for X direction in the source code fdtd2D.m. Find in fdtd2D.m the code for eq. (3.12).