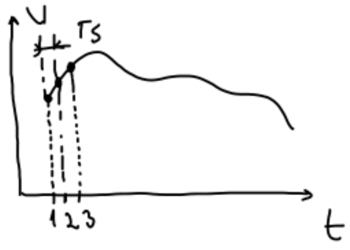
Curs 12 SIS

Filtre digitale:

$x(n)$ IN secvență de numere $\longrightarrow$ filtru digital $\longrightarrow$ $y(n)$ OUT secvență de numere



$\longrightarrow$

n	x
1	V_1
2	V_2
3	V_3
4	V_4

$f_s = \frac{1}{T_s}$ - frecvența de eșantionare („sampling frequency").

criteriul Nyquist:

$$f_s \geq 2 f_{max}$$

IN

$x(t) = A\cos(\omega t + \phi)$

după eșantionare:

$x(n) = A\cos(\omega n T_s + \phi)$, $n = 0, 1, 2 \ldots$

Funcția de transfer:

IN: $x(t) = \cos \omega t$

$\downarrow$

$x(n) = \cos(\omega n T_s)$

Vrem să determinăm ecuația diferențială a filtrului care realizează operația:

$$y(n) = x(n) + x(n-1)$$

$$y(n) = \cos(\omega n T_s) + \cos[\omega(n-1)T_s] =$$

$$= \cos(\omega n T_s) + \cos(\omega n T_s)\cos(\omega T_s) + \sin(\omega n T_s)\sin(\omega T_s)$$

$$y(n) = [1 + \cos(\omega T_s)]\cos(\omega n T_s) + \sin(\omega T_s)\sin(\omega n T_s)$$

Vrem $y(n) = |G(\omega)| \cos[\omega n T_s + \theta(\omega)]$

$G(\omega)$ - câștigul („gain")

$\theta(\omega)$ - defazajul

$$y(n) = |G(\omega)|\left[\cos(\omega n T_s)\cos\theta(\omega) - \sin(\omega n T_s)\sin\theta(\omega)\right] =$$

$$= \left[|G(\omega)|\cos\theta(\omega)\right]\cos(\omega n T_s) - \left[|G(\omega)|\sin\theta(\omega)\right]\sin\omega n T_s$$

$$|G(\omega)|\cos\theta(\omega) = 1+\cos(\omega T_s)$$

$$-|G(\omega)|\sin\theta(\omega) = \sin(\omega T_s)$$

$$-\frac{\cos\theta(\omega)}{\sin\theta(\omega)} = \frac{1+\cos(\omega T_s)}{\sin(\omega T_s)}$$

$$-\operatorname{ctg}\theta(\omega) = \frac{1+\cos(\omega T_s)}{\sin(\omega T_s)} \Rightarrow$$

$$\left[\operatorname{ctg}\frac{x}{2} = \frac{1+\cos x}{\sin x}\right]$$

$$\Rightarrow \operatorname{ctg}(-\theta(\omega)) = \operatorname{ctg}\left(\frac{\omega T_s}{2}\right)$$

$$\boxed{\theta(\omega) = -\frac{\omega T_s}{2}}$$

$$-|G(\omega)|\sin\left(-\frac{\omega T_s}{2}\right) = \sin(\omega T_s)$$

$$|G(\omega)|\cancel{\sin\left(\frac{\omega T_s}{2}\right)} = 2\cancel{\sin\left(\frac{\omega T_s}{2}\right)}\cos\left(\frac{\omega T_s}{2}\right)$$

$$\boxed{|G(\omega)| = 2\left|\cos\frac{\omega T_s}{2}\right|}$$

$$\omega = 2\pi f \; ; \; T_s = \frac{1}{f_s}$$

$$\boxed{\begin{array}{ll} |G(f)| = 2\left|\cos\left(\pi\cdot\frac{f}{f_s}\right)\right| & f_s = 2 f_{max} \\ \theta(f) = -\pi\cdot\frac{f}{f_s} & \end{array}}$$

$$y(n) = x(n) + x(n-1) \rightarrow \text{LPF}$$

Alternativ: (mai general)

$$x(n) = A e^{j(\omega n T_s + \phi)}$$

$$x(n-1) = A e^{j(\omega(n-1)T_s + \phi)}$$

$$y(n) = x(n) + x(n-1) = A e^{j(\omega n T_s + \phi)} + A e^{j(\omega(n-1)T_s + \phi)}$$

$$y(n) = A e^{j\omega n T_s} \left(1 + e^{-j\omega T_s}\right) e^{j\phi}$$

$$1 + e^{-j\omega T_s} = e^{-j\frac{\omega T_s}{2}} \underbrace{\left(e^{j\frac{\omega T_s}{2}} + e^{-j\frac{\omega T_s}{2}}\right)}_{2\cos\frac{\omega T_s}{2}} =$$

$$= 2\, e^{-j\frac{\omega T_s}{2}} \cos\frac{\omega T_s}{2}$$

$$\boxed{y(n) = 2A\cos\frac{\omega T_s}{2}\, e^{j\left[\omega n T_s + \left(\phi - \frac{\omega T_s}{2}\right)\right]}}$$

Forma generală a lui $y(n)$

$$\boxed{y(n) = A \cdot G(\omega)\, e^{j\left[\omega n T_s + (\phi + \theta(\omega))\right]}}$$

$$|G(\omega)| = 2\left|\cos\frac{\omega T_s}{2}\right|$$

$$\theta(\omega) = -\frac{\omega T_s}{2}$$

$$y(n) = \underbrace{\left[G(\omega) e^{j\theta(\omega)}\right]}_{H(e^{j\omega T_s})} \underbrace{A \cdot e^{j(\omega n T_s + \phi)}}_{x(n)}$$

H – funcția de transfer

$$\boxed{H(e^{j\omega T_s}) = G(\omega)\, e^{j\theta(\omega)}}$$

Ecuația diferențială:

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \dots + b_M x(n-M) -$$
$$- a_1 y(n-1) - a_2 y(n-2) - \dots - a_N y(n-N)$$

a_j, b_i - coeficienți de ponderare

$$\boxed{y(n) = \sum_{i=0}^{M} b_i x(n-i) - \sum_{j=1}^{N} a_j \cdot y(n-j)}$$

Filtre liniare invariante în timp
- reale: a_j, b_i - reali
- complexe: a_j, b_i - complecși.
- cauzale
- non-cauzale

Transformata „z"

- transformata Laplace discretă
- spațiul z → spațiul s discret.

$$x(n) = A e^{j(\omega n T_s + \phi)}$$
$$x(n-1) = A e^{j(\omega(n-1) T_s + \phi)} = A e^{j(\omega n T_s + \phi)} \cdot e^{-j\omega T_s} = x(n) e^{-j\omega T_s}$$
$$x(n-2) = A e^{j(\omega(n-2) T_s + \phi)} = x(n) e^{-2j\omega T_s}$$
$$\vdots$$
$$x(n-k) = x(n) e^{-j \cdot k \omega T_s}$$

$$z = e^{j\omega T_s}$$

$$x(n-k) = x(n) z^{-k}$$

$$\boxed{X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}}$$

$$y(n) = x(n) + x(n-1)$$

Transf. z:

$$Y(z) = x(n) + x(n) z^{-1} = x(n)(1 + z^{-1})$$

Proprietățile transformatei „z":

$X(z) = Z(x)$

·) $Z(\alpha \cdot x_1(n) + \beta x_2(n)) = \alpha Z(x_1(n)) + \beta Z(x_2(n)) =$
$= \alpha X_1(z) + \beta X_2(z)$

·) $Z(x(n-M)) = z^{-M} \cdot Z(x(n)) = z^{-M} \cdot X(z)$

ex: $y(n) = x(n) + x(n-1)$

transf. „z"

$$Z(y(n)) = Z(x(n) + x(n-1)) = Z(x(n)) + Z(x(n-1)) =$$
$$= Z(x(n)) + z^{-1} \cdot Z(x(n)) = (1 + z^{-1}) Z(x(n))$$

$$Y(z) = (1 + z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + z^{-1} = 1 + e^{-j\omega T_s} = 2\cos\frac{\omega T_s}{2} \cdot e^{-j\frac{\omega T_s}{2}}$$

În general:

$$y(n) = b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M) -$$
$$- a_1 y(n-1) - a_2 y(n-2) - \dots - a_N y(n-N)$$

$$Y(z) = X(z)(b_0 + b_1 z^{-1} + \dots + b_M z^{-M}) -$$
$$- Y(z)(a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N})$$

$$Y(z)(1 + a_1 z^{-1} + \dots + a_N z^{-N}) = X(z)(b_0 + b_1 z^{-1} + \dots + b_M z^{-M})$$

$$\boxed{\frac{Y(z)}{X(z)} = H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}}$$

Filtre în serie $H(z) = H_1(z) \cdot H_2(z)$
în paralel $H(z) = H_1(z) + H_2(z)$

$$\begin{cases} |H(z)| = G(\omega) \\ \theta(\omega) \end{cases}$$

p - poli
q - zerouri

$$H(z) = b_0 \cdot \frac{1 + \frac{b_1}{b_0} z^{-1} + \ldots + \frac{b_M}{b_0} z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

not. $b_0 = g$

$\frac{b_i}{b_0} = \beta_i$

$$H(z) = g \cdot \frac{1 + \beta_1 z^{-1} + \ldots + \beta_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} =$$

$$= g \cdot \frac{z^{-M}}{z^{-N}} \cdot \frac{z^M + \beta_1 z^{M-1} + \ldots + \beta_M}{z^N + a_1 z^{N-1} + \ldots + a_N}$$

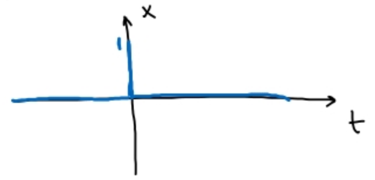
$$H(z) = g \cdot \frac{z^{-M}}{z^{-N}} \cdot \frac{(z - q_1)(z - q_2) \cdots (z - q_M)}{(z - p_1)(z - p_2) \cdots (z - p_N)}$$

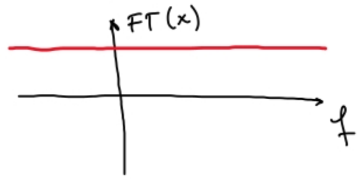
p - poli
q - zerouri

$$G(\omega) = |H(z)| = |g| \cdot \frac{|e^{j\omega T_s} - q_1||e^{j\omega T_s} - q_2| \cdots |e^{j\omega T_s} - q_M|}{|e^{j\omega T_s} - p_1| \cdots |e^{j\omega T_s} - p_N|}$$

$$\theta(\omega) = (N - M)\omega T_s + \angle(e^{j\omega T_s} - q_1) + \ldots + \angle(e^{j\omega T_s} - q_M) -$$
$$- \angle(e^{j\omega T_s} - p_1) - \ldots - \angle(e^{j\omega T_s} - p_N)$$

Răspunsul la impuls unitar:





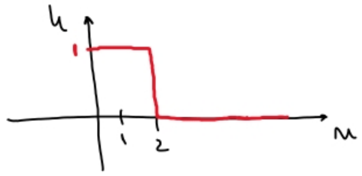
Ex: $y(n) = x(n) + x(n-1)$

$$x = \delta \rightarrow \boxed{y(n)} \rightarrow y = h$$

$h(n) \rightarrow$ răspunsul filtrului la impuls unitar

IN:
$x(0) = 1$
$x(1) = 0$
$x(2) = 0$.
$\vdots$
$x(j) = 0$

$h(0) = x(0) + x(-1) = 1$
$h(1) = x(1) + x(0) = 0 + 1 = 1$
$h(2) = x(2) + x(1) = 0 + 0 = 0$



filtru stabil $\rightarrow h(n) \rightarrow 0, \; n \rightarrow \infty$

Probleme cu filtre digitale:

$H(z) \to G(\omega), \theta(\omega)$, tipul filtrului, ordinul filtrului

P1. $y(n) = x(n) + x(n-1)$

– transformata „z":

$$\mathcal{Z}(y(n)) = \mathcal{Z}(x(n) + x(n-1))$$

$$Y(z) = X(z) + z^{-1}X(z) = (1+z^{-1})X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + z^{-1}$$

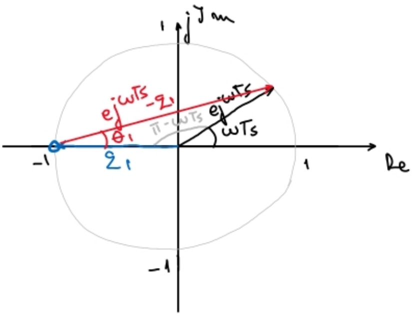
$$\boxed{H(z) = z^{-1}\cdot(z+1)}$$

$z^{-1} \Rightarrow$ filtrul are ordin 1
un zero
nici un pol

$z + 1 = 0 \Rightarrow z = \boxed{-1 = q_1}$

$$G(\omega) = |H(z)| = |z^{-1}(z+1)| = \underbrace{|z^{-1}|}_{1}\,|z+1| \qquad z = e^{j\omega T_s}$$

Metoda grafică



⌜T. lui Pitagora generalizată⌝
$c^2 = a^2 + b^2 - 2ab\cos\alpha$

$$c^2 = a^2 + b^2 - 2ab\cos(\pi - \omega T_s) =$$
$$= a^2 + b^2 - 2ab(\underbrace{\cos\pi}_{-1}\cos\omega T_s + \underbrace{\sin\pi}_{0}\sin\omega T_s) =$$
$$= a^2 + b^2 + 2ab\cos\omega T_s$$

$$c = \sqrt{a^2 + b^2 + 2ab\cos\omega T_s}$$

$c = |e^{j\omega T_s} - q_1|$

$a = |q_1| = 1$

$b = |e^{j\omega T_s}| = 1$

$$|e^{j\omega T_s} - q_1| = \sqrt{2 + 2\cos\omega T_s} = \sqrt{2\underbrace{(1+\cos\omega T_s)}_{2\cos^2\frac{\omega T_s}{2}}} = \sqrt{4\cos^2\frac{\omega T_s}{2}}$$

$$G(\omega) = |H(z)| = \left|\sqrt{4\cos^2\frac{\omega T_s}{2}}\right| = \left|\pm 2\cos\frac{\omega T_s}{2}\right| = 2\cos\frac{\omega T_s}{2}$$

$$H(z) = \underbrace{z^{-1}}_{\theta_0}\cdot\underbrace{(z+1)}_{\theta_1}$$

$$\theta(\omega) = \theta_0 + \theta_1$$

$$z^{-1} = e^{-j\omega T_s} = e^{j\theta_0}$$

$$\theta_0 = -\omega T_s$$

$$2\theta_1 + \cancel{\pi} - \omega T_s = \cancel{\pi}$$

$$2\theta_1 = \omega T_s$$

$$\theta_1 = \frac{\omega T_s}{2}$$

$$\theta(\omega) = -\omega T_s + \frac{\omega T_s}{2} = -\frac{\omega T_s}{2}$$

P2. $y(n) = x(n) + x(n-1) + y(n-1)$

- transf. z:

$$Y(z) = X(z) + z^{-1}X(z) + z^{-1}Y(z)$$

$$Y(z)(1-z^{-1}) = X(z)(1+z^{-1})$$

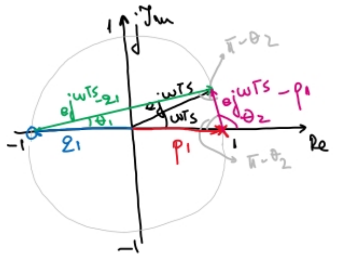
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-z^{-1}}$$

$$H(z) = \frac{\cancel{z^{-1}}(z+1)}{\cancel{z^{-1}}(z-1)}$$

$$H(z) = \frac{z+1}{z-1}$$

- ord. 1
- un zero $\longrightarrow z+1=0 \Rightarrow z=-1=z_1$
- un pol $\longrightarrow z-1=0 \Rightarrow z=1=p_1$

$$H(z) = \frac{z-z_1}{z-p_1} \Rightarrow G(\omega) = |H(z)| = \frac{|e^{j\omega T_s} - z_1|}{|e^{j\omega T_s} - p_1|}$$



$|e^{j\omega T_s} - z_1| = 2\left|\cos\frac{\omega T_s}{2}\right|$

$\theta_1 = \frac{\omega T_s}{2}$

$$c^2 = a^2 + b^2 - 2ab\cos\omega T_s$$

$$c = \sqrt{1+1-2\cos\omega T_s} = \sqrt{2\underbrace{(1-\cos\omega T_s)}_{2\sin^2\frac{\omega T_s}{2}}} = \sqrt{4\sin^2\frac{\omega T_s}{2}}$$

$$|e^{j\omega T_s} - p_1| = 2\left|\sin\frac{\omega T_s}{2}\right|$$

$$G(\omega) = \frac{\cancel{2}\left|\cos\frac{\omega T_s}{2}\right|}{\cancel{2}\left|\sin\frac{\omega T_s}{2}\right|} = \left|\mathrm{ctg}\,\frac{\omega T_s}{2}\right| = \left|\mathrm{ctg}\,\pi\cdot\frac{f}{f_s}\right|$$

$$\theta(\omega) = \theta_1 - \theta_2$$

$$\omega T_s + 2\pi - 2\theta_2 = \pi$$

$$-2\theta_2 = -\pi - \omega T_s$$

$$\theta_2 = \frac{\pi}{2} + \frac{\omega T_s}{2}$$

$$\theta(\omega) = \cancel{\frac{\omega T_s}{2}} - \frac{\pi}{2} - \cancel{\frac{\omega T_s}{2}}$$

$$\theta(\omega) = -\frac{\pi}{2}$$

Temă: analizați următorul filtru digital:
$y(n) = x(n-1) + x(n-2) + y(n-1)$.